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Chapter -6

Waves in the Atmosphere

The mathematical solution of most problems of dynamic meteorology is exceedingly difficult owing to the nonlinear nature of the terms in the equation of motion. So many of the problems in meteorology can be solved only when they are linearized. The motion that is to be studied can often be treated as a small perturbation super imposed on an undisturbed state of the atmosphere. Such problems arise, for instance, in the theory of the origin of tropical cyclones. According to V.Bjerknes, cyclones develop as small wave perturbations at the boundary between two air masses of different density and velocity. If such a wave is unstable, its amplitude increases and a cyclone develop. The undisturbed motion here is assumed as geostrophic motion.

6.1. The perturbation equations:

For deriving the perturbation equations, the following assumptions are made:

i. Both the total and undisturbed motions satisfy the equation of motion.
ii. The perturbation quantities

Let us consider the instantaneous motion ($\bar{u}$) as the sum of undisturbed (U) plus the perturbed motion (u'). Writing like this:

$\bar{u} = U + u'$ $\quad\bar{v} = V + v'$ $\quad\bar{w} = W + w'$ and $\quad\bar{p} = P + p'$ ....(6.1)

The total equation of motion can be written as

$$\frac{d\bar{u}}{dt} = \frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\alpha\frac{\partial \bar{p}}{\partial x} + fv$$

$$\frac{d\bar{v}}{dt} = \frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \bar{w}\frac{\partial \bar{v}}{\partial z} = -\alpha\frac{\partial \bar{p}}{\partial y} - fu \quad \text{........................(6.2)}$$

$$\frac{d\bar{w}}{dt} = \frac{\partial \bar{w}}{\partial t} + \bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{v}\frac{\partial \bar{w}}{\partial y} + \bar{w}\frac{\partial \bar{w}}{\partial z} = -\alpha\frac{\partial \bar{p}}{\partial z} - g$$

And the equation of continuity is

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \quad \text{...(6.3)}$$

According to the assumption (i) the same equations (6.2) and (6.3) are satisfied for undisturbed motion also. Which means they are obtained by simply replacing the capital letters.

But to find the perturbed motion (6.1) are to be substituted in (6.2) and expanded.
Let us consider first the x equation:

$$\frac{\partial}{\partial t}(U+u')+(U+u')\frac{\partial}{\partial x}(U+u')+(V+v')\frac{\partial}{\partial y}(U+u')+(W+w')\frac{\partial}{\partial z}(U+u')=-\alpha\frac{\partial}{\partial x}(P+p')+f(V+v')$$

……(6.4)

For undisturbed motion we can write:

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}=-\alpha\frac{\partial P}{\partial x}+fV \quad \text{……………….(6.5)}$$

To get perturbed motion subtract (6.5) from (6.4) and neglect the product of perturbations.

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+V\frac{\partial u'}{\partial y}+W\frac{\partial u'}{\partial z}+u'\frac{\partial U}{\partial x}+v'\frac{\partial U}{\partial y}+w'\frac{\partial U}{\partial z}=-\frac{1}{\rho}\frac{\partial p'}{\partial x}+fv' \quad \text{……………(6.6a)}$$

Similarly we can write y and z component equations as:

$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}+V\frac{\partial v'}{\partial y}+W\frac{\partial v'}{\partial z}+u'\frac{\partial V}{\partial x}+v'\frac{\partial V}{\partial y}+w'\frac{\partial V}{\partial z}=-\frac{1}{\rho}\frac{\partial p'}{\partial y}-fu' \quad \text{…………(6.6b)}$$

$$\frac{\partial w'}{\partial t}+U\frac{\partial w'}{\partial x}+V\frac{\partial w'}{\partial y}+W\frac{\partial w'}{\partial z}+u'\frac{\partial W}{\partial x}+v'\frac{\partial W}{\partial y}+w'\frac{\partial W}{\partial z}=-\frac{1}{\rho}\frac{\partial p'}{\partial z} \quad \text{…………….(6.6c)}$$

The equation of continuity also can be written as:

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0 \quad \text{……………………….(6.6d)}$$

These equations are called perturbation equations.

6.2. Acoustic or sound waves:

Sound waves are longitudinal waves. The particle oscillations in longitudinal waves are parallel to the direction of propagation. Sound is propagated by the alternating adiabatic compression and expansion of the medium.
To obtain one dimensional sound waves with the help of perturbation method, let us consider that the waves propagate in a straight pipe parallel to the x-axis.
To exclude the interference of transverse oscillations (particle motions at right angles to the direction of propagation), let us assume that meridional and vertical velocities are zero.
So v = w = 0 and u = u(x,t) and doesn't depend on y and z coordinates.
Then the equations of momentum, continuity and thermodynamic energy for adiabatic motions are as follows:

$$\frac{du}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \quad \text{………………………….(6.1)}$$

$$\frac{1}{\rho}\frac{d\rho}{dt}+\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]=0.$$

As 3rd and 4th terms are zero for transverse oscillations, equation (6.2) reduces to

$$\frac{1}{\rho}\frac{d\rho}{dt}+\left[\frac{\partial u}{\partial x}\right]=0\text{.......(6.2)}$$

And the thermodynamic energy equation can be written as

$$\frac{d}{dt}(\ln\theta)=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.3)$$

(Please note under adiabatic conditions θ is conserved). Equation (6.3) can be modified taking the potential temperature equation

$$\theta=T\left[\frac{p_0}{p}\right]^{R/C_p}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.4)$$

Where p_0 is suface pressure =1000mb. We may eliminate θ in equation (6.3) using (6.4).

Taking logs in equation (6.4)

$$\ln\theta=\ln T+\frac{R}{c_p}\ln\left(\frac{1000}{p}\right)=\ln T-\frac{R}{c_p}\ln(p)\ \ldots\ldots..(6.5)$$

Using P = RρT equation (6.5) can be written as

$$\ln\theta=\ln P\left(1-\frac{R}{c_p}\right)-[\ln(R)+\ln(\rho)]=\frac{1}{\gamma}\ln p-\ln\rho\ \text{ since }\ 1-\frac{R}{c_p}=\frac{c_v}{c_p}=\frac{1}{\gamma}$$

Or differentiating with respect to 't'

$$\frac{d}{dt}\ln\theta=\frac{1}{\gamma}\frac{d}{dt}\ln p-\frac{d}{dt}\ln\rho=0\ \ldots\ldots\ldots.(6.6)$$

Eliminating ρ between (6.2) and (6.6), we can get

$$\frac{1}{\gamma}\frac{d}{dt}\ln p-\left(-\frac{\partial u}{\partial x}\right)=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(6.7)$$

Applying perturbation technique using the identities

$$u_{(x,t)}=\bar{u}+u',\qquad p_{(x,t)}=\bar{p}+p',\qquad \rho_{(x,t)}=\bar{\rho}+\rho'\ \ldots\ldots\ldots\ldots\ldots(6.8)$$

Substitute eqn (6.8) in (6.1) first after expansion. Then we can get

$$\frac{\partial}{\partial t}\left(\bar{u}+u'\right)+\left(\bar{u}+u'\right)\frac{\partial}{\partial x}\left(\bar{u}+u'\right)+\frac{1}{\left(\bar{\rho}+\rho'\right)}\frac{\partial}{\partial x}\left(\bar{p}+p'\right)=0\ \ldots\ldots\ldots(6.9)$$

Substituting (6.8) in (6.7)

First rewrite eqn (6.7) as $\frac{1}{\gamma}\frac{d}{dt}\ln p + \left(\frac{\partial u}{\partial x}\right) = 0$

$\frac{1}{\gamma}\frac{1}{p}\frac{dp}{dt} + \left(\frac{\partial u}{\partial x}\right) = 0$ or $\frac{dp}{dt} + \gamma p\left(\frac{\partial u}{\partial x}\right) = 0$

Or $\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\left(\frac{\partial u}{\partial x}\right) = 0$ ……………………..(6.10)

Substituting (6.8) in (6.10) we can write

$$\frac{\partial}{\partial t}(\bar{p}+p') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{p}+p') + \gamma(\bar{p}+p')\frac{\partial}{\partial x}(\bar{u}+u') = 0 \quad \text{…………….(6.11)}$$

For expanding eqn (6.9) , the Bousinesq approximation may be adopted:

$$\frac{1}{\bar{\rho}+\rho'} = \frac{1}{\bar{\rho}\left(1+\frac{\rho'}{\bar{\rho}}\right)} = \frac{1}{\bar{\rho}}\left[\left(1+\frac{\rho'}{\bar{\rho}}\right)^{-1}\right] = \frac{1}{\bar{\rho}}\left[\left(1-\frac{\rho'}{\bar{\rho}}\right)\right] = \frac{1}{\bar{\rho}} \qquad \left(\because \frac{\rho'}{\bar{\rho}}<1\right)$$

The equation (6.9) can be written as

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial u'}{\partial x} + u'\frac{\partial \bar{u}}{\partial x} + u'\frac{\partial u'}{\partial x} + \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} = 0 \quad \text{…………(6.12)}$$

We know the undisturbed flow as

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} = 0 \quad \text{………………………(6.13)}$$

Subtracting (6.13) from (6.12) and neglecting the nonlinear terms associated with perturbation terms like $u'\frac{\partial \bar{u}}{\partial x} = u'\frac{\partial u'}{\partial x} = 0$ we get

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} = 0 \quad \text{………………………(6.14)}$$

Similarly we can write (6.11) as

$$\frac{\partial p'}{\partial t} + \bar{u}\frac{\partial p'}{\partial x} + \gamma\,\bar{p}\frac{\partial u'}{\partial x} = 0 \quad \text{………………………..(6.15)}$$

To find the solutions of (6.14) and (6.15) let us assume there exists a wave of the form

$u' = Ae^{ik(x-ct)}$ and $p' = Be^{ik(x-ct)}$ ………(6.16)

where k = 2π/L is the wave number. If (6.16) are the solutions of (6.14) and (6.15), they should be satisfied when they are substituted in them. So differentiate (6.16) and substitute in (6.14) and (6.15) and simplify.

Then the equation (6.14) becomes:

$$A\left(\bar{u}-c\right)=-\frac{B}{\bar{\rho}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.17)$$

Similarly (6.15) becomes:

$$B\left(\bar{u}-c\right)=-\gamma\,\bar{p}\,A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.18)$$

Eliminating A and B from (6.17) and (6.18) we get

$$B\left(\bar{u}-c\right)=-\gamma\,\bar{p}\left[-\frac{B}{\bar{\rho}\left(\bar{u}-c\right)}\right]$$

$$\left(\bar{u}-c\right)^2=\frac{\gamma\,\bar{p}}{\bar{\rho}} \;=\; \gamma R T \qquad (\because P=R\rho T)$$

Or $C=\bar{u}\pm\sqrt{(\gamma RT)}$ ………………….(6.19)

Equation (6.19) says that the speed of propagation relative to zonal current $\bar{u}$ is $\sqrt{(\gamma RT)}$. This quantity is called the adiabatic speed of sound. This is also called *Lamb* wave.
The mean zonal velocity here plays only a role of Doppler shifting the sound wave so that the frequency

$$\eta = KC = K\{\bar{u}\pm\sqrt{(\gamma RT)}\}$$

Corresponding to a given wave number K appears higher to an observer in the down stream from the source than to an observer in the upstream.

6.3. Gravity Waves:

Consider a homogeneous incompressible fluid. The undisturbed flow consists of a constant horizontal velocity U between a lower, horizontal boundary (z = 0) and an upper free surface whose undisturbed position is z = h. the waves are assumed to have infinite lateral extent in the y direction and thus are essentially in x-z direction. The friction and coriolis forces are neglected.
With the above assumptions, we can say U = const, V = W = 0

And $-\frac{1}{\rho}\frac{\partial p}{\partial z}=g$ such that $0\le z\le h$

Assumptions:

1. The wave lie in the xz plane
2. The coriolis and frictional forces neglected
3. The bottom of the layer is a rigid earth's surface
4. the undisturbed flow is zonal (U = const, V=W=0)
5. The waves have infinite lateral extent

Due to these assumptions the perturbation equations turned out as

Then we can write the perturbation equations (6.6 a,c and d) as

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} \quad \ldots\ldots\ldots(6.20)$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} \quad \ldots\ldots\ldots.(6.21)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.22)$$

Assume that the solution of equations (6.20) to (6.22) are as

$$\begin{aligned} u' &= \psi(z)e^{ik(x-ct)} \\ w' &= \phi(z)e^{ik(x-ct)} \quad \ldots\ldots\ldots\ldots\ldots..(6.23) \\ p' &= \delta(z)e^{ik(x-ct)} \end{aligned}$$

To solve the equations substitute (6.23) in equations (6.20) to (6.22) and on simplification and apply the boundary conditions:

a) The kinematic boundary condition says that the vertical velocity (w) vanish at the bottom (z) i.e. on $z = 0$, w = 0

b) The second dynamic boundary condition used is that the total pressure (P+p') is zero at the top surface. In other words at the free surface $\frac{d}{dt}(P + p') = 0$.

Then we can get:

$$(U - C) = \pm\left[\frac{g}{k}\tanh(kz)\right]^{1/2} \quad \text{or} \quad C = U \pm \left[\frac{g}{k}\tanh(kz)\right]^{1/2} \ldots\ldots\ldots(6.24)$$

As z = h at the top of the free atmosphere, on substitution

$C = U \pm \left[\frac{g}{k}\tanh(kh)\right]^{1/2}$, putting k =2π/L we get

$$C = U \pm \left[\frac{gL}{2\pi}\tanh\left(\frac{2\pi h}{L}\right)\right]^{1/2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.25)$$

The first term on the right hand side of equation (6.25) is called convective term and the second term is called the dynamic term. The positive and negative signs tell about the positive and negative x directions of propagation of the wave.

Two special cases can be discussed here.

Case (i) : Deep water waves:

If $h > 0.4$ L, tan h $(2\pi h/L) \to 1$

So Equation (6.25) becomes $C = U \pm \left[\frac{gL}{2\pi}\right]^{1/2}$ ………………(6.26)

These waves are called deep water or short waves.

Case (ii): Shallow water waves:

If $h < L/25$, then tan h $(2\pi h/L) \to 2\pi h/L$

Then equation (6.25) becomes $C = U \pm \sqrt{gh}$ ……………..(6.27)

These are also called long waves.

For homogeneous atmosphere, $h = P_0/\rho g = 7.991$ km, where $P_0/\rho = RT_0$. Then equation (6.27) turns out to $C = U \pm \sqrt{RT_0}$. This is nothing but equation (6.19) which is called Newtonian speed of sound and is equal to 280 m/s at 0^0C.

Note that the dynamic term of equation 6.25 is real and so the gravity waves are neutral.

6.4. Atmospheric Rossby Waves:

Rossby waves are very important for large scale meteorological processes. These are called planetary waves because they are very long waves encircling the whole globe.

In a barotropic atmosphere these waves conserve absolute vorticity ($\zeta + f$) which owes its existence to the variation of Coriolis parameter with latitude called the β effect.

Rossby wave propagation can be understood in a qualitative manner by considering a closed chain of fluid parcels initially aligned along a circle of latitude.

From the potential vorticity theorem $[\frac{d}{dt}\left(\frac{\zeta+f}{D}\right) = 0]$, consider the flow of fluid at constant depth (D) on a latitudinal circle. On this initial latitudinal circle assume that $\zeta = 0$ at initial time t_0. Let suppose at time t_1, δy is the meridional displacement of a fluid parcel from the initial latitude. Then at t_1 we can write the absolute vorticity as:

$$(\zeta + f)_{t1} = (\zeta + f)_{t0}$$

$$(\zeta)_{t1} + (f)_{t1} = (\zeta)_{t0} + (f)_{t0}$$

$$(\zeta)_{t1} + (f)_{t1} = (f)_{t0} \quad \text{since } (\zeta)_{t0} = 0$$

Or $(\zeta)_{t1} = -(f)_{t1} + (f)_{t0} = -\{(f)_{t1} - (f)_{t0}\} = -\left(\frac{df}{dy}\right)\delta y = \beta\delta y$ ……….. (1)

The minus sign shows that the initial value is less and final value is more (as coriolis force increases with increase of latitude).

From equation (1), it is evident that if the chain of parcels is subject to a sinusoidal meridional displacement then the CAV is positive (cyclonic) for a southward displacement and negative (anticyclonic) for a northward displacement as shown in figure below. This means the pattern of vorticity maxima and minima propagates westward. This westward propagating vorticity field constitutes Rossby waves.

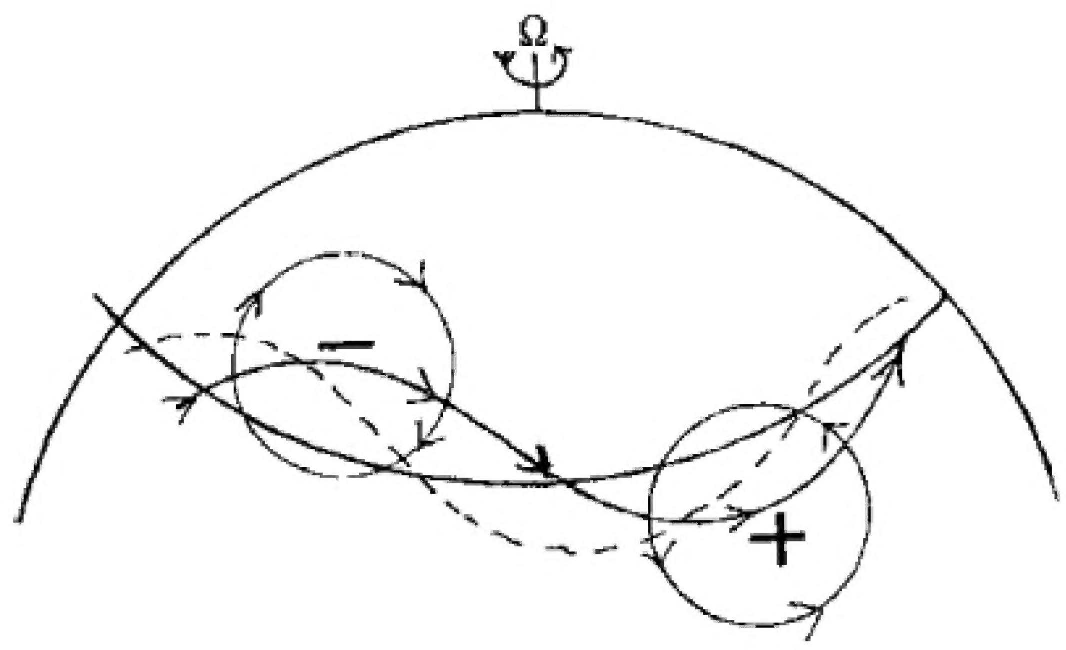

Fig. CAV trajectories and westward propagating Rossby waves.

The solid curve without arrows is the latitude circle. The dashed wave shows the original perturbation. The solid wave with arrow marks shows the west ward displacement of the pattern due to advection. The arrow marks on the solid wave shows the cyclonic and anitcyclonic motions and not the direction of wave propagation. Note that the positive circle is at a lower latitude than the negative circle implies the wave is westward propagating.

Derivation of Rossby waves:

Assumptions:

1. The flow is horizontal (zonal) and non divergent
2. The atmosphere is autobarotropic so that basic mean current ($\bar{u}$) is constant with height and along meridian.
3. The perturbation components u', v' and p' are independent of y. That is they have infinite lateral extent.

Consider the barotropic vorticity equation: $\frac{d}{dt}(\zeta + f) = 0$ ……………………(2)

Or $\frac{d\zeta}{dt} + \frac{df}{dt} = 0$

On the XY- plane (z component is not considered) we can write:

$$\left[\frac{\partial\zeta}{dt} + u\,\frac{\partial\zeta}{\partial x} + v\,\frac{\partial\zeta}{\partial y}\right] + \left[\frac{\partial f}{dt} + u\,\frac{\partial f}{\partial x} + v\,\frac{\partial f}{\partial y}\right] = 0 \quad \text{………………….. (3)}$$

Since $\frac{\partial f}{dt} = 0 \; and \; u\,\frac{\partial f}{\partial x} = 0$, the equation (3) reduc es to

$$\left[\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y}\right]\zeta + \left[v\,\frac{\partial f}{\partial y}\right] = 0 \quad \text{…………………………….. (4)}$$

Applying the perturbation technique, keeping the above assumptions in view:

$$u = \bar{u} + u' \;, \quad v = v' \;, \qquad \zeta = \zeta' \qquad \text{………………………(5)}$$

such that $\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}$, $u' = -\frac{\partial \psi}{\partial y}$, $v' = \frac{\partial \psi}{\partial x}$ …………………(6)

where ψ is stream function.

So $\zeta' = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi$ …………………(7a)

$\frac{\partial \zeta'}{\partial t} = \frac{\partial}{\partial t}(\nabla^2 \psi)$ …………………(7)

Substituting perturbations (eqn 5) in equation (4)

$$\left[\frac{\partial}{\partial t} + (\bar{u} + u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right]\zeta' + \left[v'\frac{\partial f}{\partial y}\right] = 0$$

Expanding $\frac{\partial \zeta'}{\partial t} + \bar{u}\frac{\partial \zeta'}{\partial x} + u'\frac{\partial \zeta'}{\partial x} + v'\frac{\partial \zeta'}{\partial y} + v'\frac{\partial f}{\partial y}$

Substituting equations (6) and (7) in the above equation, we can write as

$\frac{\partial}{\partial t}(\nabla^2 \psi) + \bar{u}\frac{\partial \zeta'}{\partial x} + \left(-\frac{\partial \psi}{\partial y}\right)\frac{\partial \zeta'}{\partial x} + \left(\frac{\partial \psi}{\partial x}\right)\frac{\partial \zeta'}{\partial y} + \left(\frac{\partial \psi}{\partial x}\right)\frac{\partial f}{\partial y} = 0$ …………………(8)

Note the third and fourth terms gets cancelled. Use eqn (7a) for (ζ') in second term and $\frac{\partial f}{\partial y} = \beta$. Then the above equation (8) can be written as:

$\frac{\partial}{\partial t}(\nabla^2 \psi) + \bar{u}\frac{\partial}{\partial x}(\nabla^2 \psi) + \beta\left(\frac{\partial \psi}{\partial x}\right) = 0$ …………………(9)

Let the solution of this second order differential equation (9) is

$\psi = \psi_0 e^{i(kx+ly-vt)}$ …………………(10)

then find $\nabla^2 \psi$ and substitute in eqn (9). To find $\nabla^2 \psi$, double differentiate equation (10)

$\frac{\partial \psi}{\partial x} = \psi_0 (ik)\, e^{i(kx+ly-vt)}$ …………………(10 a)

Similarly,

$\frac{\partial^2 \psi}{\partial x^2} = \psi_0 (ik)(ik)\, e^{i(kx+ly-vt)} = -\psi_0 (k^2)\, e^{i(kx+ly-vt)}$ since $i^2 = -1$. …………………(11)

Similarly, $\frac{\partial^2 \psi}{\partial x^2} = -\psi_0 (l^2)\, e^{i(kx+ly-vt)}$ …………………(12)

Combining equations (11) and (12), $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} = -\psi_0\, e^{i(kx+ly-vt)}(k^2 + l^2)$

Using equation (10), $\nabla^2 \psi = -(k^2 + l^2)\,\psi$ …………………(13)

Substituting eqn (13) in (9), also use equation (10)

$\frac{\partial}{\partial t}[-(k^2 + l^2)\,\psi\,] + \bar{u}\frac{\partial}{\partial x}[-(k^2 + l^2)\,\psi\,] + \beta\,\psi_0 (ik)\, e^{i(kx+ly-vt)} = 0$ …………………(14)

Using again equation (10) we can write (14) as

$\frac{\partial}{\partial t}[-(k^2 + l^2)\,\psi_0 e^{i(kx+ly-vt)}\,] + \bar{u}\frac{\partial}{\partial x}[-(k^2 + l^2)\,\psi_0 e^{i(kx+ly-vt)}\,] + \beta\,\psi_0 (ik)\, e^{i(kx+ly-vt)} = 0$ …………………(15)

Solving equation (15) by differentiating first term w.r.t 't' and second term w.r.t 'x' and replace third term using eqn (10a)

$-(k^2 + l^2)(-iv)\,\psi_0 e^{i(kx+ly-vt)} + \bar{u}\,[-(k^2 + l^2)(ik)\,\psi_0 e^{i(kx+ly-vt)}\,] + \beta\frac{\partial \psi}{\partial x} = 0$

Again replace ψ using eqn (10)

$$- (k^2 + l^2)(-iv)\,\psi_0 e^{i(kx+ly-vt)} + \bar{u}\left[-(k^2+l^2)(ik)\,\psi_0 e^{i(kx+ly-vt)}\right] + \beta \frac{\partial}{\partial x}\left[\psi_0 e^{i(kx+ly-vt)}\right] = 0$$

Differentiate again w.r.t 'x'.

$$- (k^2 + l^2)(-iv)\,\psi_0 e^{i(kx+ly-vt)} + \bar{u}\left[-(k^2+l^2)(ik)\,\psi_0 e^{i(kx+ly-vt)}\right] + \beta(ik)\,\psi_0 e^{i(kx+ly-vt)} = 0$$

Divide throughout by $(i)\,\psi_0 e^{i(kx+ly-vt)}$

$$(k^2+l^2)(v) + \bar{u}\left[-(k^2+l^2)(k)\right] + \beta(k) = 0$$

$$(k^2+l^2)(v - \bar{u}\,k) + \beta\,k = 0$$

Or $(v - \bar{u}\,k) = -\frac{\beta\,k}{(k^2+l^2)}$ or $(v) = \bar{u}\,k - \frac{\beta\,k}{(k^2+l^2)}$ or $\frac{v}{k} = \bar{u} - \frac{\beta}{(k^2+l^2)} = c$

Or $\bar{u} - c = \frac{\beta}{(k^2+l^2)}$ …………………(16)

In the equation (16), k is zonal wave number and 'l' is the meridional wave number. If k $= \frac{2\pi}{\lambda}$, where λ is the zonal wave length and $l = \frac{1}{d}$, where 'd' is the width of the wave in the meridian, we can write:

$$\bar{u} - c = \frac{\beta}{(k^2+l^2)} = \frac{\beta}{\left(\left[{4\pi^2}/{\lambda^2}\right] + l^2\right)}$$

If 'l' is zero ($d = \infty$), then we can write $\bar{u} - c = \frac{\beta\,\lambda^2}{4\pi^2}$

or $c = \bar{u} - \frac{\beta\,\lambda^2}{4\pi^2}$ …………………(17)

This equation (17) is called Rossby equation for the speed of long waves of infinite lateral extent ($d = \infty$) in the westerlies (minus sign) over mean zonal flow ($\bar{u}$). In the equation (17), $\bar{u}$ denotes the mean velocity and $-\frac{\beta\,\lambda^2}{4\pi^2}$ is the variation over the mean flow.

Thus the Rossby wave propagates westward relative to the mean zonal flow in the mid tropospheric height. Further the Rossby wave speed depends on the zonal (k) and meridional (l) wave numbers. Therefore Rossby waves are dispersive waves whose phase speeds increase with wave length.

For a typical midlatitude synoptic disturbance (say hurricane) with zonal wave length (K) of 6000 km and latitude width (d) of 3000 km, the Rossby wave speed (c) relative to the mean zonal flow ($\bar{u}$) from equation (17) is approximately (– 6 m/s). This means synoptic scale Rossby waves move slowly.

If the frequency of Rossby waves is compared with other waves from the table given below:

S.No	Type of wave	Waves per day
1.	Sound wave	10^8
2.	Long gravitatinal wave	10^4
3.	Inertia & shearing	10^3
4.	cyclone	10^1

5.	Rossby	10^{-1}

From the table, in the case of a cyclone, the frequency is 10 waves per day whereas in the case of Rossby waves ($10^{-1} = 1/10$) one wave occurs in 10 days.